

TIT-HEP-293
HUTP-95/A018
May 1995

Quantum Gravity is Renormalizable near Two Dimensions

Yoshihisa KITAZAWA^{*}

*Department of Physics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo 152, Japan*[†]

and

*Lyman Laboratory of Physics, Harvard University,
Cambridge, MA 02138, USA*

Abstract

We prove the renormalizability of quantum gravity near two dimensions. The successful strategy is to keep the volume preserving diffeomorphism as the manifest symmetry of the theory. The general covariance is recovered by further imposing the conformal invariance. The proof utilizes BRS formalism in parallel with Yang-Mills theory. The crucial ingredient of the proof is the relation between the conformal anomaly and the β functions.

^{*}E-mail address : kitazawa@phys.titech.ac.jp and kitazawa@huhepl.harvard.edu

[†]Permanent address and address after May 29

1 Introduction

In two dimensions, Einstein-Hilbert action is well known to be topological. Therefore the Einstein tensor identically vanishes in two dimensions. On the other hand, the β function of the gravitational coupling constant G indicates that the theory is asymptotically free[1]. The β function of the gravitational coupling constant in $D = 2 + \epsilon$ dimensions at the one loop level is[2, 3]

$$\mu \frac{\partial}{\partial \mu} G = \epsilon G - \frac{25 - c}{24\pi} G^2, \quad (1)$$

where μ is a renormalization scale and c counts the matter contents. It shows that the theory is well defined at short distance as long as $c < 25$. The short distance fixed point of the β function is $G^* = 24\pi\epsilon/(25 - c)$.

We can take the two dimensional limit of this fixed point if we let $c \rightarrow 25$ simultaneously with arbitrary strength of G^* . This is the critical string. If $c < 25$, the gravitational coupling constant grows at long distance as $8\pi/G \sim Q^2 \log(\mu)$ in two dimensions where $Q^2 = (25 - c)/3$. Therefore we need the cosmological constant Λ to define the theory. The scaling dimension of the cosmological constant is $\Lambda \sim (1/\mu)^{Q\alpha}$, where $\alpha = \frac{Q}{2}(1 - \sqrt{1 - \frac{8}{Q^2}})$ [7, 8, 3, 4]. Hence we naturally explain the double scaling relation of the matrix models[9]:

$$\exp(-8\pi/G) \sim \Lambda^{\frac{Q}{\alpha}}. \quad (2)$$

However the theory is not always controllable in this way. For $1 < c < 25$, α becomes complex and the infrared behavior of the theory appears to be too wild.

Therefore somewhat mysterious asymptotic freedom of the topological gravitational coupling constant fits our knowledge of string theory very nicely. If we take the asymptotic freedom seriously, we may contemplate the possibility that consistent quantum gravity exists beyond two dimensions. The $2 + \epsilon$ dimensional expansion of quantum gravity enables us to study such an idea systematically. The pocket of a new phase with width ϵ opens up for $0 < G < G^*$ in quantum gravity beyond two dimensions. In this weak coupling phase physical theories with $1 < c < 25$ are well defined since the infrared behavior of the theory is trivial. Furthermore this phase resembles our universe.

However the difficulty of this program has been recognized in [3] which may be rephrased in the following way. Let us treat the conformal mode of the metric as a matter field. We also need to introduce a reference metric. From such a view point, the general covariant theory always possesses the conformal invariance with respect to the reference metric. However it is also well known that it is not possible to maintain the conformal invariance in quantum field theory with nontrivial β functions. Therefore we are inescapable from the presence of the anomaly in the crucial symmetry of the theory.

It has been proposed to only keep the volume preserving diffeomorphism as the manifest symmetry. The general covariance can be recovered by further imposing the conformal invariance with respect to the reference metric on the theory including the quantum effect[4, 5]. A possible proof of this proposal was suggested in [5] and it was checked by an explicit calculation at the two loop level[6]. In this paper we prove the renormalizability of quantum gravity near two dimensions through this approach. We thereby lay the foundation for the $2 + \epsilon$ dimensional expansion of quantum gravity. We hope that this approach also sheds light on the lattice approach[10, 11].

In section two, we recall the formulation of the quantum gravity in $2 + \epsilon$ dimensions. We set up the BRS formalism and derive the Ward-Takahashi identities. In section three, we solve the WT identity to determine the bare action. We give the inductive proof of the renormalizability in section four. We show that the divergences of the theory can be canceled by the counter terms which can be supplied by the bare action. We conclude in section five with discussions.

2 BRS Invariance and Ward-Takahashi Identity

In this section, we recall the formulation of quantum gravity in $2 + \epsilon$ dimensions. We then set up the BRS formalism and derive Ward-Takahashi identities. We decompose the metric into the conformal factor and the rest as $g_{\mu\nu} = \hat{g}_{\mu\rho}(e^h)^\rho_\nu e^{-\phi} = \tilde{g}_{\mu\nu} e^{-\phi}$. $\hat{g}_{\mu\nu}$ is a background metric. The tensor indices of fields are raised and lowered by the background metric. $h_{\mu\nu}$ is a traceless symmetric tensor. We have decomposed the metric into two different types of

variables since their renormalization properties are very different[3, 4, 5].

We consider the following generic action:

$$I = \int \frac{\mu^\epsilon}{G} \{ \tilde{R} \Phi(X) + \frac{1}{2} \tilde{g}^{\mu\nu} G_{ij}(X) \partial_\mu X^i \partial_\nu X^j \}, \quad (3)$$

where $\int = \int d^D x \sqrt{\tilde{g}}$ denotes the integration over the D dimensional spacetime. In this expression, X^i are $N = c+1$ copies of real scalar fields. The conformal mode of the metric is treated as one of them. \tilde{R} is the scalar curvature made out of $\tilde{g}_{\mu\nu}$. G and the wave function renormalization of X^i fields are fixed by requiring $\Phi(0) = 1$ and $G_{ij}(0) = \eta_{ij}$ where η_{ij} is the flat metric in N dimensions. Generic theories with general covariance can be described in this way by further imposing the conformal invariance with respect to the background metric. Our approach is therefore a natural generalization of the nonlinear sigma model approach in string theory [12, 13, 14] into higher dimensions.

The crucial symmetry of the theory is the invariance under the following gauge transformation:

$$\begin{aligned} \delta \tilde{g}_{\mu\nu} &= \partial_\mu \epsilon^\rho \tilde{g}_{\rho\nu} + \tilde{g}_{\mu\rho} \partial_\nu \epsilon^\rho + \epsilon^\rho \partial_\rho \tilde{g}_{\mu\nu} - \frac{2}{D} \partial_\rho \epsilon^\rho \tilde{g}_{\mu\nu}, \\ \delta X^i &= \epsilon^\rho \partial_\rho X^i - (D-1) G^{ij} \frac{\partial \Phi}{\partial X^j} \frac{2}{D} \partial_\rho \epsilon^\rho. \end{aligned} \quad (4)$$

In order to prove the renormalizability of the theory, we set up the BRS formalism. The BRS transformation of these fields δ_B is defined by replacing the gauge parameter by the ghost field $\epsilon^\mu \rightarrow C^\mu$. The BRS transformation of $h_{\mu\nu}$ field is defined through the relation $\tilde{g} = \hat{g} e^h$. The BRS transformation of ghost, antighost and auxiliary field is

$$\begin{aligned} \delta_B C^\mu &= C^\nu \partial_\nu C^\mu, \\ \delta_B \bar{C}^\mu &= \lambda^\mu, \\ \delta_B \lambda^\mu &= 0. \end{aligned} \quad (5)$$

The BRS transformation can be shown to be nilpotent $\delta_B^2 = 0$.

Our proof proceeds in parallel with Yang-Mills theory case[15]. However the conformal anomaly forces us further investigations. We denote $A_i = (h_{\mu\nu}, X_i)$. We also introduce a

gauge fixing function $F_\alpha(A)$. It is an arbitrary function of A with dimension one. The gauge fixed action is

$$S = I + \int \left[-\frac{G}{2\mu^\epsilon} \lambda_\alpha \lambda^\alpha + \lambda^\alpha F_\alpha - \bar{C}^\alpha \delta_B F_\alpha - K^i \delta_B A_i + L_\alpha \delta_B C^\alpha \right]. \quad (6)$$

Here we have introduced sources K and L for the composite operators. The criterion for the action I is the invariance under the volume preserving diffeomorphism. Hence the BRS invariance is broken if there exists conformal anomaly:

$$\delta_B I = \int T_\alpha^\alpha \frac{2}{D} \partial_\beta C^\beta, \quad (7)$$

where $T_\alpha^\alpha = -\hat{g}_{\mu\nu} \delta I / \delta \hat{g}_{\mu\nu}$.

The partition function is

$$Z = e^W = \int [dA dC d\bar{C} d\lambda] \exp(-S + \int [J^i A_i + \bar{\eta}_\alpha C^\alpha + \bar{C}^\alpha \eta_\alpha + l_\alpha \lambda^\alpha]). \quad (8)$$

By the change of the variables with BRS transformation form, we obtain the Ward-Takahashi identity for the generating functional of the connected Green's functions:

$$\int (J^i \frac{\delta}{\delta K^i} + \bar{\eta}_\alpha \frac{\delta}{\delta L_\alpha} + \eta_\alpha \frac{\delta}{\delta l_\alpha}) W = < \int T_\alpha^\alpha \frac{2}{D} \partial_\beta C^\beta >. \quad (9)$$

The WT identity for the effective action is obtained by the Legendre transformation:

$$\int \left[\frac{\delta \Gamma}{\delta A_i} \frac{\delta \Gamma}{\delta K^i} + \frac{\delta \Gamma}{\delta C^\alpha} \frac{\delta \Gamma}{\delta L_\alpha} - \lambda^\alpha \frac{\delta \Gamma}{\delta \bar{C}^\alpha} \right] = - \int T_\alpha^\alpha \frac{2}{D} \partial_\beta C^\beta. \quad (10)$$

In order to make the above expression finite, we need to add all necessary counter terms to S . The bare action S^0 obtained in this way satisfies the same equation

$$\int \left[\frac{\delta S^0}{\delta A_i} \frac{\delta S^0}{\delta K^i} + \frac{\delta S^0}{\delta C^\alpha} \frac{\delta S^0}{\delta L_\alpha} - \lambda^\alpha \frac{\delta S^0}{\delta \bar{C}^\alpha} \right] = - \int T_\alpha^\alpha \frac{2}{D} \partial_\beta C^\beta. \quad (11)$$

On the other hand, eq. (9) follows from eq. (11) in dimensional regularization. Therefore the following famous relation between the trace anomaly and the bare action holds as an operator identity: $T_\alpha^\alpha = -\hat{g}_{\mu\nu} \delta I^0 / \delta \hat{g}_{\mu\nu}$. To simplify notations, we introduce an auxiliary field M_α and add to the action the combination $-\int M_\alpha \lambda^\alpha$ in such a way that $\lambda^\alpha = -\frac{\delta \Gamma}{\delta M_\alpha} = -\frac{\delta S}{\delta M_\alpha}$. Then the left hand side of eq. (10) and eq. (11) become homogeneous quadratic equations which we write symbolically as $\Gamma * \Gamma$ and $S^0 * S^0$.

3 Analysis of the Bare Action

In this section, we solve eq. (11) to determine S^0 . In the subsequent considerations, we deal with the bare fields and the bare BRS transformation. The bare BRS transformation is given in eq. (4) and eq. (5) in terms of the bare fields. The precise relation between the bare and the renormalized quantities will be explained in the next section. S^0 can always be decomposed into the part I^0 which involves only A_i fields and the rest. The right hand side of eq. (11) is determined by I^0 only.

Let us examine the general structure of the bare action. By power counting, it has to be a local functional of fields and sources with dimension D . We also have the ghost number conservation rule and its ghost number has to be zero. By these dimension and ghost number considerations, it is easy to see that K and L appear only linearly in S^0 :

$$S^0 = \int [-K^i(\delta'_B A_i) + L_\alpha(\delta'_B C^\alpha)] + \tilde{S}, \quad (12)$$

where δ'_B denotes most general BRS like transformations with correct dimension and ghost number. It is also easy to see that there are no λ and hence no \bar{C} dependence in δ'_B . Since λ has dimension $1 + \epsilon$, \tilde{S} can be at most quadratic in λ :

$$\tilde{S} = \int \left[-\frac{G^0}{2} \tilde{E}_{\alpha\beta} \lambda^\alpha \lambda^\beta + \lambda^\alpha \tilde{F}_\alpha + \tilde{L} \right], \quad (13)$$

where $\tilde{E}_{\alpha\beta}$ and \tilde{F}_α are general functions of A, C and \bar{C} with dimension zero and one respectively. G^0 is the bare gravitational coupling constant and it is the only quantity with dimension $-\epsilon$.

In order to determine the structure of the bare action, we decompose $S^0 = S + \delta S$ and $I^0 = I + \delta I$. We may assume without loss of generality that δS and δI are small. General solutions may be obtained by integrating these solutions. Then we obtain the following equation for δS :

$$\int \Delta \delta S = - \int T^\alpha_\alpha \frac{2}{D} \partial_\beta C^\beta, \quad (14)$$

where the trace anomaly on the right hand side is that of δI . Here we have introduced a

differential operator:

$$\Delta = \frac{\partial S}{\partial A_i} \frac{\partial}{\partial K^i} + \frac{\partial S}{\partial K^i} \frac{\partial}{\partial A_i} + \frac{\partial S}{\partial C^\alpha} \frac{\partial}{\partial L_\alpha} + \frac{\partial S}{\partial L_\alpha} \frac{\partial}{\partial C^\alpha} + \frac{\partial S}{\partial M_\alpha} \frac{\partial}{\partial \bar{C}^\alpha}. \quad (15)$$

We denote below by θ^i the set of all anticommuting fields $K^i, C^\alpha, \bar{C}^\alpha$ and x_i all commuting fields A_i, L_α, M_α . Under the following infinitesimal change of the variables

$$\begin{aligned} x'_i &= x_i - \frac{\partial \psi}{\partial \theta^i}, \\ (\theta^i)' &= \theta^i + \frac{\partial \psi}{\partial x_i}, \end{aligned} \quad (16)$$

the action $S(\theta^i, x_i)$ changes as

$$S(\theta', x') - S(\theta, x) = \Delta \psi. \quad (17)$$

We also have the following relation:

$$\Delta^2 = \left[-\frac{\partial}{\partial \theta^j} (S * S) \right] \frac{\partial}{\partial x_j} + \left[\frac{\partial}{\partial x_j} (S * S) \right] \frac{\partial}{\partial \theta^j}. \quad (18)$$

The most general solution for δS which satisfies eq. (14) is

$$\delta S = \int \delta_B(\bar{C}^\alpha (F'_\alpha + G^0 \lambda^\beta E'_{\alpha\beta})) + \delta I(A), \quad (19)$$

where $\delta I(A)$ is invariant under the volume preserving diffeomorphism. E' and F' are general functions of A, C and \bar{C} with dimension zero and one respectively.

The BRS exact part can be associated with a canonical transformation on the fields. It can be understood as a freedom in association with the gauge fixing procedure. Therefore we conclude that the bare action is similar to the tree level action in terms of the bare fields with arbitrariness in association with the gauge fixing procedure.

4 Inductive Proof of the Renormalizability

In this section, we give the inductive proof of the renormalizability of quantum gravity by the $2 + \epsilon$ dimensional expansion approach. We will show that all necessary counter terms can be

supplied from the bare action which is invariant under the volume preserving diffeomorphism. We expand the trace anomaly by the power series of G as $T_\alpha^\alpha = \sum_{i=0}^\infty \beta_i G^i$. Our strategy is to tune it to be $O(G^l)$ at l loop level. Note that we have assumed it starts at $O(1)$. We need to fine tune the tree action I for this purpose to start with since it is $O(1/G)$ in general.

Our analysis is based on a loopwise expansion of the effective action:

$$\Gamma = \sum_{l=0}^\infty \Gamma_l, \quad (20)$$

in which Γ_0 is the tree level action S . We assume as an induction hypothesis that we have been able to construct an action S_{l-1}^0 with counter terms which satisfies eq. (11) and renders Γ finite at $l-1$ loop order. Then the right hand side of eq. (11) is proportional to the trace anomaly.

Let the couplings and the operators dual to them as $\{G, \lambda_k\}$ and $\{\tilde{R}, \Lambda^k\}$. In our action, the couplings are $\{G, \Phi-1, G_{ij}-\eta_{ij}\}$. λ_k represents two arbitrary functions of X^i fields and hence is equivalent to the infinite numbers of coupling constants. The bare couplings are

$$\begin{aligned} \frac{1}{G^0} &= \frac{\mu^\epsilon}{G} Z_G = \mu^\epsilon \left(\frac{1}{G} - \sum_{\nu=1}^{l-1} \frac{a_\nu^\nu}{\epsilon^\nu} \right), \\ \lambda_k^0 &= \lambda_k + \sum_{\nu=1}^{l-1} \frac{a_k^\nu}{\epsilon^\nu}. \end{aligned} \quad (21)$$

Here we have $1/\epsilon^{l-1}$ poles at most at $l-1$ loop order. The β functions follow as

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} G &= \epsilon G + \beta_G \\ &= \epsilon G - a_0^1 G^2 - G^3 \frac{\partial}{\partial G} a_0^1, \\ \mu \frac{\partial}{\partial \mu} \lambda_k &= \beta_{\lambda_k} = -G \frac{\partial}{\partial G} a_k^1. \end{aligned} \quad (22)$$

As it is well known, only the residues of the simple pole in ϵ contribute to the β functions. The coefficients of the higher poles in the bare couplings are determined by the finiteness of the β functions (pole identity).

The bare action is

$$I^0 = \int \left[\frac{1}{G^0} \tilde{R}^0 + \frac{1}{2G^0} \eta_{ij} \partial_\mu X_0^i \partial_\nu X_0^j \tilde{g}^{\mu\nu} + \frac{\lambda_k^0}{G^0} \Lambda_0^k \right]. \quad (23)$$

The trace anomaly of the bare action is

$$\begin{aligned} & \frac{1}{G^0}(\epsilon\Phi^0 + 2(D-1)\partial^i\Phi^0\partial_i\Phi^0)\tilde{R} \\ & + \frac{1}{2G^0}(\epsilon G_{ij}^0 + 4(D-1)\nabla_i\partial_j\Phi^0)\partial_\mu X_0^i\partial_\nu X_0^j\tilde{g}^{\mu\nu}. \end{aligned} \quad (24)$$

We need to rewrite this expression in terms of the renormalized operators[14]. The wave function renormalization can be ignored since we are investigating the renormalization of the operators after the wave function renormalization. It can be justified by using the equations of motion in dimensional regularization. By the same reason we can do away with the total derivatives in the trace anomaly.

We have shown that the bare action consists of the BRS exact part and the part which is invariant under the volume preserving diffeomorphism. We assume as an inductive hypothesis that we have been able to renormalize the theory up to $l-1$ loop level with counter terms which can be supplied by the bare action. Namely the counter terms themselves consist of the BRS exact part and the part invariant under the volume preserving diffeomorphism. Therefore we assume that the operator mixing has occurred only within each sector, namely the BRS exact operators and the operators which are invariant under the volume preserving diffeomorphism. For this reason we can ignore the BRS exact operators in the following considerations.

We introduce the renormalized operators which are defined as

$$\begin{aligned} \tilde{R} &= (-G^2\frac{\partial}{\partial G} - G\lambda_k\frac{\partial}{\partial\lambda_k})I^0, \\ \Lambda_k &= G\frac{\partial}{\partial\lambda_k}I^0. \end{aligned} \quad (25)$$

This operator is finite up to $l-1$ loop order by our inductive assumption. We fix the wave function renormalization of X^i fields as $\frac{1}{G^0}\eta_{ij}\partial_\mu X_0^i\partial_\nu X_0^j\tilde{g}^{\mu\nu} = \frac{\mu^\epsilon}{G}\eta_{ij}\partial_\mu Y^i\partial_\nu Y^j\tilde{g}^{\mu\nu}$. For this purpose we rewrite $\sqrt{Z_G}X_0^i = Y^i$ and do not differentiate Y^i in eq. (25). We also subtract the already finite kinetic term for X^i fields from \tilde{R} .

We can expand the trace anomaly eq. (24) in terms of the renormalized couplings and

operators. The trace anomaly is

$$\begin{aligned} & \frac{1}{G} \left\{ \epsilon \Phi + \frac{\beta_G}{G} - \beta_\Phi + \frac{\beta_G}{G} \left(1 - \frac{1}{2} X^i \frac{\partial}{\partial X^i} \right) (\Phi - 1) + 2(D-1) \partial^i \Phi \partial_i \Phi \right\} \tilde{R} \\ & + \frac{1}{2G} \left\{ \epsilon G_{ij} - \beta_{G_{ij}} - \frac{\beta_G}{G} \frac{1}{2} X^k \frac{\partial}{\partial X^k} G_{ij} + 4(D-1) \nabla_i \partial_j \Phi \right\} \partial_\mu X^i \partial_\nu X^j \tilde{g}^{\mu\nu}. \end{aligned} \quad (26)$$

In this way, the trace anomaly can be expressed by the β functions and the renormalized operators. The singularities in ϵ cancel out up to $O(G^{l-2})$ by the assumption. We can tune the couplings in the theory such that the conformal anomaly vanishes up to $O(G^{l-2})$. Now we have tuned the coupling constants of the theory such that the the right hand side of eq. (11) is $O(G^{l-1})$. At this order, the trace anomaly has divergences in general. However these divergences are the inevitable consequences of the lower loop divergences. Let us add the counter terms at l loop order which are required by the pole identity to make β functions finite. These counter terms with higher poles in ϵ are invariant under the volume preserving diffeomorphism. In this way we can make the right hand side of eq. (11) finite at l loop order.

Now we can solve eq. (10) at $O(G^{l-1})$ to obtain:

$$S * \Gamma_l + \Gamma_l * S = \Delta \Gamma_l = - \sum_{m=1}^{l-1} \Gamma_m * \Gamma_{l-m} - T_\alpha^\alpha \frac{2}{D} \partial_\beta C^\beta. \quad (27)$$

This equation determines the divergent part of Γ_l^{div} . Since we have subtracted all subdivergences, the divergences are local. The solution of this equation can be decomposed into the BRS exact part and the rest. The nontrivial part of Γ_l^{div} has to be invariant under the volume preserving diffeomorphism. Furthermore new divergences which contribute to the β functions at l loop order have to be conformally invariant in two dimensional limit. It is because the right hand side of eq. (27) has been made finite.

Γ_l^{div} may also have the BRS exact part of $\Delta\psi$ form since $\Delta^2 = 0$ to this order. The general form of ψ is:

$$\psi = K^i \Psi'_i + L_\alpha \Theta'^\alpha_\beta C^\beta + \bar{C}^\alpha (F'_\alpha + G \mu^{-\epsilon} \lambda^\beta E'_{\alpha\beta}), \quad (28)$$

where Ψ' and Θ' are general functions of A, C, \bar{C} with dimension zero and vanishing ghost number. As we have explained, the BRS exact part can be associated with a canonical

transformation on the fields. Here we consider the physical implications of these canonical transformations. Under this transformation, the part of S^0 linear in K and L changes as:

$$\begin{aligned} K^i \delta_B A_i &\rightarrow K^i \delta_B A_i + K^i \delta_B (\Psi'_i) - K^i \frac{\partial \delta_B A_i}{\partial A_j} \Psi'_j + K^i \frac{\partial \delta_B A_i}{\partial C^\alpha} \Theta'^\alpha_\beta C^\beta, \\ L_\alpha \delta_B C^\alpha &\rightarrow L_\alpha \delta_B C^\alpha - L_\alpha \delta_B (\Theta'^\alpha_\beta C^\beta) + L_\alpha \frac{\partial \delta_B C^\alpha}{\partial C^\beta} \Theta'^\beta_\gamma C^\gamma - L_\alpha \frac{\partial \delta_B C^\alpha}{\partial A_i} \Psi'_i. \end{aligned} \quad (29)$$

These infinitesimal deformations can be interpreted as the change of the functional form of the BRS transformation in association with the wave function renormalization of the fields. Note that the functional form of the BRS transformation has to change in terms of the renormalized variables, although the functional form of the BRS transformation remains the same in terms of the bare fields. The renormalized BRS transformation continues to be nilpotent. The rest of the BRS exact part causes the renormalization of the gauge fixing part.

By defining the bare action at l loop level

$$S_l^0 = S_{l-1}^0 - \Gamma_l^{div} + \text{higher orders}, \quad (30)$$

it is possible to render Γ l loop finite. Here Γ_l^{div} includes divergences with higher poles as well as simple poles in ϵ . The counter terms can be interpreted as the coupling constant and wave function renormalization of a bare action as we have explained. By doing so, we are able to construct the bare action S_l^0 which satisfies eq. (11). Now the circle is complete and we have proven the renormalizability of quantum gravity near two dimensions.

Let us consider the following model (conformal Einstein gravity) as a concrete example[5, 6]:

$$\begin{aligned} \Phi &= 1 + \sqrt{\epsilon} a \psi + \epsilon b (\psi^2 - \varphi_i^2), \\ G_{ij} &= \eta_{ij}, \end{aligned} \quad (31)$$

where η_{ij} is the flat Minkowski metric in N dimensions. In this model, the operators which are invariant under the volume preserving diffeomorphism and conformally invariant in two

dimensions are $\int \tilde{R}$, $\int [\sqrt{\epsilon} a \psi \tilde{R} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu}]$ and $\int \frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu\nu}$. These operators are invariant under the transformation eq. (4) up to $O(\epsilon)$.

Therefore at l loop level, new divergences of the following form may arise:

$$\frac{\mu^\epsilon}{G} \int \left[\frac{a^1 G}{\epsilon} \tilde{R} - \frac{z^1}{\epsilon} (\sqrt{\epsilon} a \psi \tilde{R} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu}) + \frac{z'}{\epsilon} (\frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu\nu}) \right]. \quad (32)$$

The last divergence in the above expression can be taken care of by the wave function renormalization of φ_i . The second part of these divergences can be dealt with by the following wave function renormalization: $\psi \rightarrow (1 + \frac{z^1}{2\epsilon})\psi + \frac{a}{4\sqrt{\epsilon}b} \frac{z^1}{\epsilon}$. By these wave function renormalizations, we can supply the required counter terms from the original action. Since b is associated with an explicit factor of ϵ , this procedure introduces a finite counter term of $\mu^\epsilon \int (\psi^2 - \varphi_i^2) \tilde{R}$ type. The β function of b receives no contribution. In order to cancel the remaining divergence, we need the counter term $-\mu^\epsilon \int (a_0^1/\epsilon) \tilde{R}$ where $a_0^1 = a^1 + z^1 a^2/4bG$.

Therefore this model is renormalizable to all orders with the following bare action:

$$\frac{\mu^\epsilon}{G} \int [Z_G \tilde{R} + \sqrt{\epsilon} a \psi \tilde{R} + \epsilon b (\psi^2 - \varphi_i^2) \tilde{R} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu} + \frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu\nu}]. \quad (33)$$

The β functions of b vanishes while $\mu \frac{\partial}{\partial \mu} G = \epsilon G Z_G / (1 - G \frac{\partial}{\partial G}) Z_G$ agrees with eq. (22). The trace anomaly eq. (26) becomes

$$\begin{aligned} & \frac{1}{G} \{ \epsilon Z_G / (1 - G \frac{\partial}{\partial G}) Z_G + \epsilon (\Phi - 1) + 2(D-1) \partial^i \Phi \partial_i \Phi \} \tilde{R} \\ & + \frac{1}{2G} \{ \epsilon \eta_{ij} + 4(D-1) \partial_i \partial_j \Phi \} \partial_\mu X^i \partial_\nu X^j \tilde{g}^{\mu\nu} \end{aligned} \quad (34)$$

It is finite as long as the β function of G is finite. The trace anomaly vanishes to all orders if

$$\begin{aligned} b &= \frac{1}{8(D-1)}, \\ \epsilon G + \beta_G &= 2(D-1) a^2 G \epsilon. \end{aligned} \quad (35)$$

The short distance fixed point of the renormalization group where $a = \epsilon G + \beta_G = 0$ is certainly consistent with eq. (35). Since the Einstein action is recovered for large ψ , it may be sufficient to construct the theory at the fixed point. Here we draw the analogy

with the spontaneous symmetry breaking in field theory. The theory with nonvanishing a can be obtained from the fixed point theory by giving the vacuum expectation values to $\psi \rightarrow \psi + a/(2\sqrt{\epsilon}b)$. a can be determined by the second equation of eq. (35). The renormalization group evolution may be viewed as such a symmetry breaking process and the renormalizability of the theory on the whole renormalization group trajectory naturally follows in such an interpretation.

5 Conclusions and Discussions

In this paper we have constructed a proof of the renormalizability of quantum gravity near two dimensions. We thereby lay the foundation for the $2 + \epsilon$ dimensional expansion of quantum gravity. We have proven that all necessary counter terms can be supplied by the bare action which is invariant under the volume preserving diffeomorphism. We can systematically cancel the trace anomaly to all orders by tuning the coupling constants of the theory. In particular the Einstein gravity with conformally coupled scalar fields is shown to be renormalizable by tuning the gravitational coupling constant to all orders.

In this proof we have assumed that the dimensional regularization preserves all important symmetries of the theory. The Jacobian in association with the change of the variables which involves no derivatives is always trivial in dimensional regularization. We also need to assume some infrared regularization. A gauge invariant regularization is to consider a closed finite universe. However the simplest possibility is to add a mass term to $h_{\mu\nu}$ field[6]. Although it breaks the BRS invariance, we can show that such a soft breaking of the symmetry will not spoil the renormalizability of the theory.

We need to modify the action in the following way:

$$S \rightarrow S + \int [\frac{\mu^\epsilon}{4G} m^2 h_{\mu\nu} h^{\mu\nu} + m^2 \bar{C}_\alpha C^\alpha + \tilde{M} \delta_B (\frac{\mu^\epsilon}{4G} h_{\mu\nu} h^{\mu\nu} + \bar{C}_\alpha C^\alpha)], \quad (36)$$

where \tilde{M} is an external source. The additional BRS exact part can be absorbed by the original action by the canonical transformation of the external sources. The WT identity

eq. (10) for the effective action is modified as follows:

$$\Gamma * \Gamma = -T^\alpha_\alpha \frac{2}{D} \partial_\beta C^\beta - m^2 \frac{\partial \Gamma}{\partial \tilde{M}}. \quad (37)$$

The eq. (27) which determines the divergence of Γ_l is also modified as

$$S * \Gamma_l + \Gamma_l * S = \Delta \Gamma_l = - \sum_{m=1}^{l-1} \Gamma_m * \Gamma_{l-m} - T^\alpha_\alpha \frac{2}{D} \partial_\beta C^\beta - m^2 \frac{\partial \Gamma_l}{\partial \tilde{M}}. \quad (38)$$

The general divergence structure which is allowed by this equation is

$$\frac{\mu^\epsilon}{G} \int [m^2 f(A, C, \bar{C}) + \tilde{M} \delta_B f] + \tilde{\Gamma}_l^{div}, \quad (39)$$

where $\tilde{\Gamma}_l^{div}$ is obtained from Γ_l^{div} in section four by the canonical transformation of the sources. f is a general function of A, C and \bar{C} with dimension zero. Through this analysis we have shown that the renormalization property of the dimension two operators remains intact while the soft symmetry breaking term may be renormalized into a general dimension zero function.

Finally we make a comment on the Unitarity of the theory. Let us assume that we are in the weak coupling phase. The only poles in the Green's functions arise at $p^2 = 0$ in the gravity sector. However the theory flows to classical Einstein gravity at long distance and hence there should be no problem with Unitarity in the theory.

I am grateful to E. D'Hoker, D. Gross and C. Vafa for their hospitality at UCLA, Princeton and Harvard University respectively where part of this work has been carried out. I also appreciate discussions on this subject with H. Kawai, M. Ninomiya, T. Aida, J. Nishimura, A. Tsuchiya and A. Migdal. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

References

- [1] S. Weinberg, in General Relativity, an Einstein Centenary Survey, eds. S.W. Hawking and W. Israel (Cambridge University Press, 1979).
R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. **B133** (1978) 417.
S.M. Christensen and M.J. Duff, Phys. Lett. **B79** (1978) 213.
- [2] H. Kawai and M. Ninomiya, Nucl. Phys. **B336** (1990) 115.
- [3] H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. **B393**(1993) 280.
- [4] H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. **B404** (1993) 684.
- [5] T. Aida, Y. Kitazawa, H. Kawai and M. Ninomiya, Nucl. Phys. **B427** (1994) 158.
- [6] T. Aida, Y. Kitazawa, J. Nishimura and A. Tsuchiya, KEK-TH-423, TIT-HEP-275, UT-Komaba/94-22, to appear in Nucl.Phys. B.
- [7] A.M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893.
V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.
- [8] F. David, Mod. Phys. Lett. **A3** (1988) 1651.
J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 504.
- [9] D.J. Gross and A.A. Migdal, Phys. Rev. Lett. **64** (1990) 127; M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635; E. Brezin and V. Kazakov, Phys. Lett. **B236** (1990) 144.
- [10] M.E. Agishtein and A.A. Migdal, Mod.Phys.Lett. **A7** (1992) 1039.
- [11] J. Ambjorn and J. Jurkiewicz, Phys.Lett. **B278** (1992) 42.
- [12] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. **B262** (1985) 593.
- [13] E.S. Fradkin and A.A. Tseytlin, Nucl Phys. **B261** (1985) 1.

- [14] A.A. Tseytlin, Nucl. Phys. **B294** (1987) 383.
- [15] For a review see J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford Univ. Press, 1989) sect. 21.